Short Note on Geometrical Engineering and Topological String

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Geometric Engineering of 4d $\mathcal{N}=2$ theories

In this section we show that how geometric engineering of 4d $\mathcal{N}=2$ theory works in general, to sum up in one sentence: geometric engineering is the top to bottom construction of gauge theories. To be detailed, we first have a string theory or M theory in 10/11 dimensions, then we change the internal manifold to change the low dimensional theory's property. Historically this is regarded as string compactification, but since compactness is not necessarily required (as if we need to throw gravity away while preserving supersymmetry, a non-compact Calabi-Yau is needed), a general name of geometric engineering is better.

Symmetry from Geometry

We can do the engineering dimension by dimension, first on a 4d Calabi-Yau, which is generally a K3 surface. However, since we dislike compactness at the moment, we choose the ALE space which locally describes the singularity, globally noncompact, and still Calab-Yau, to be our choice.

The ALE space is generally (where G is a finite subgroup of $SU(2)$ which preserves the holomorphic 2-form which ensures this is a Calabi-Yau)

$$\mathbb{C}^2/G \quad + \quad \textit{resolution} \tag{1}$$

which the resolution stand for resolving the orbifold singularity, which replace the singularity by some $\mathbb{P}^1 = S^2$, this depends on the discrete group.

Symmetry from Geometry

For $G = \mathbb{Z}_n$ as a example, one use $n-1$ such $\mathbb{P}^1$ (denoted $C_j, j = 1, \cdots, n-1$) and they are nontrivially intersected

$$\begin{aligned} C_i \cap C_i &= -2 \\ C_i \cap C_j &= 1 \quad if|i-j| = 1 \\ C_i \cap C_j &= 0 \quad if|i-j| > 1 \end{aligned} \tag{2}$$

which is beautifully related to the Cartan matrix (also, Dynkin diagram) of $A_{n-1} = su(n)$, as a realization of the McKay correspondence.

Moreover, all simple laced Lie algebra (A,D,E) can be so constructed, for $D_{n-1} = so(2n-2)$ it is by (the double cover of) dihedral(2) group of n elements, and E_6, E_7, E_8 by (the double cover of) tetrahedral(4), octahedral(8) and dodecahedral(12) group.

Symmetry from Geometry

Then we do the compactification on a Riemann surface Σ_g to reduce the dimension to 4, the general statement is the number of adjoint hypermultiplets equals to the genus

$$\#(\text{4d adjoint hypermultiplets}) = g \tag{3}$$

the origin of these hypermultiplets can be roughly understood by starting with the gauge theory in $d = 6$ and compactifying it on Σ_g, since $H_1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^{2g}$, we get $2g$ complex scalars from one 6d vector, which is in the 4d hypermultiplet in adjoint representation.

$\mathbb{C}^2/G \times T^2$

For $\mathbb{T}^2$ is a Calabi-Yau, the 6d $\mathcal{N} = 2$ reduce to 4d $\mathcal{N} = 4$ in this moment, there's also

$$\mathrm{vol}(T^2) = \frac{1}{g_{YM}^2} \tag{4}$$

where the T-duality of the torus becomes Montonen-Olive duality in 4d $\mathcal{N} = 4$ SYM.

$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{P}^1$

Which is the case that we care the most, since the result is the Seiberg-Witten theory without hypermultiplets. This model is also known as the local $\mathbb{P}^1 \times \mathbb{P}^1$, we have two Kähler parameter t_b, t_f here, one of which is identified to the mass of W bosons and the other to Yang-Mills coupling

$$\begin{aligned} m_W &= t_f \\ t_b &= 1/g_{YM}^2 \end{aligned} \tag{5}$$

$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{P}^1$

And one obvious but important thing is, for generic value of g_{YM}^2 and fibre moduli, the string theory on such background has more information then the gauge theory, only when the field theory limit

$$t_f \to 0, \quad t_b \to \infty, \quad t_b = -\log t_f \tag{6}$$

the string theory goes back to the field theory. This limit sends the string scale to infinity. If we do not take this decoupling limit , we get a theory which includes information about compactification on S^1 from five to four dimensions, 4d instantons are 5d loop-running particles in this aspect.

Hypermultiplet, Seiberg-Witten Curve

We have already know the adjoint hypermultiplet, but what about matter in other representations, especially the fundamental representation of the gauge group? These are singularity enhancement of the fibre[12], for example the singularity is enhanced from A_{n-1} to A_n at one point, this means locally the "gauge group" is $SU(n+1)$ instead of $SU(n)$, however, this suffers a symmetry breaking which makes it to $SU(n) \times U(1)$, which is recognized as a matter with charge localized to a point of the base Σ_g. Obviously, the breaking pattern of such enhanced "gauge symmetry" determines the representation of the matter hypermultiplet.

This leads to two result, the first is that this singularity enhancement equals to a blow up of the fibre, so the number of blow up we take is the number of matters. The next is that the localized matter can be viewed as a puncture, so $\Sigma_{g,n}$ also means n matters.

Hypermultiplet, Seiberg-Witten Curve

Breaking Patterns	Lie Group	Representations
$A_n \to A_{n-k} \times A_{k-1}$	$SU(k) \times SU(n-k+1) \times U(1)$	$(\mathbf{k}, \mathbf{n-k+1})$
$D_n \to D_{n-1}$	$SO(2n-2) \times SO(2)$	$\mathbf{2n-2}$
$D_n \to A_{n-1}$	$SU(n) \times U(1)$	$\mathbf{n(n-1)/2}$
$D_n \to D_{n-r} \times A_{r-1}$	$SO(2n-2r) \times SU(r) \times U(1)$	$(\mathbf{2n-2r}, \mathbf{r})$ $(\mathbf{1}, \mathbf{r(r-1)/2})$
$E_6 \to D_5$	$SO(10) \times U(1)$	$\mathbf{16}$
$E_6 \to A_5$	$SU(6) \times U(1)$	$\mathbf{20}$
$E_7 \to D_6$	$SO(12) \times U(1)$	$\mathbf{32}$
$E_7 \to E_6$	$E_6 \times U(1)$	$\mathbf{27}$
$E_7 \to A_6$	$SU(7) \times U(1)$	$\mathbf{35} \oplus \mathbf{7}$
$E_8 \to E_7$	$E_7 \times U(1)$	$\mathbf{56}$

Hypermultiplet, Seiberg-Witten Curve

This leads to two results, the first is that this singularity enhancement equals to a blow up of the fibre, so the number of blow up we take is the number of matters. The next is that the localized matter can be viewed as a puncture, so $\Sigma_{g,n}$ also means n matters.

Moreover, it is proved that for local mirror symmetry models, the mirror curve pf the Calabi-Yau is equivalent to the Seiberg-Witten curve from engineered Seiberg-Witten model. So another understanding is that blow up the local Calabi-Yau changes the mirror curve and the Seiberg-Witten curve, making it from $N_f = 0$ to $N_f = 1$.

Topological Strings and the Engineering

The topological string computes particular F-terms in the effective action which involve the vector multiplets. These terms can be written conveniently in terms of the $\mathcal{N}=2$ Weyl multiplet, which is a chiral superfield $\mathcal{W}_{\alpha\beta}$ with lowest component $F_{\alpha\beta}$ (which is the graviphoton field $F_{\mu\nu}=F_{\alpha\beta}(\gamma_\mu)^{\alpha\dot{\sigma}}(\gamma_\nu)^{\beta}_{\dot{\sigma}}+F_{\dot{\alpha}\dot{\beta}}(\gamma_\mu)^{\dot{\alpha}}_{\sigma}(\gamma_\nu)^{\dot{\beta}\sigma}$). For combining

$$\mathcal{W}^2=\mathcal{W}_{\alpha\beta}\mathcal{W}_{\alpha'\beta'}\epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'} \tag{7}$$

and the term is written as, where X^I are the vector superfields

$$\int d^4x\int d^4\theta F_g(X^I)(\mathcal{W}^2)^g \tag{8}$$

where the physical and topological strings connected by

$$(F_g(X^I))_{phys}=(F_g)_{top} \tag{9}$$

expanding this in components result

$$\int d^4 x F_g(X^I)(R^2 F^{2g-2}) \tag{10}$$

which is the gravitational correction to amplitude of graviphotons, Gopakumar and Vafa integrated this out by introducing the M-theory (or 5d) description by the Schwinger method, where the Gopakumar-Vafa invariants appears.

Topological Strings and the Engineering

Since topological string is necessarily defined merely on the Calabi-Yau threefold and the 4d gauge theory is defined by geometric engineered from a Calabi-Yau threefold, the natural question is how these two theories connect to each other. Thanks to Nekrasov's contribution to the nonperturbative effects of the gauge theory side and lots of tricks we can apply to the topological string side, enables studies to reveal the connections of these two theories.

Nekrasov's Conjecture

We have already know Nekrasov's Instanton Partition function[1][2], which is a sum that encodes the $G \times T^2$ equivariant cohomology of the moduli space $\tilde{\mathcal{M}}_k$

$$Z(a, \epsilon_1, \epsilon_2; q) = 1 + \sum_{k=1}^{\infty} q^k Z_k = \sum_{k=1}^{\infty} q^k \oint_{\tilde{\mathcal{M}}_k} 1 \tag{11}$$

generally it depends on a function $\mathcal{F}^{inst}(a, \epsilon_1, \epsilon_2; q)$ that is analytic in ϵ_1, ϵ_2 near $\epsilon_1 = \epsilon_2 = 0$

$$Z(a, \epsilon_1, \epsilon_2; q) = exp\left(\frac{\mathcal{F}^{inst}(a, \epsilon_1, \epsilon_2; q)}{\epsilon_1 \epsilon_2}\right) \tag{12}$$

Nekrasov's Conjecture

theres also the expression for Z in the case $\epsilon_1 = -\epsilon_2 = \hbar$ for $G = SU(N)$

$$Z(a, \hbar, -\hbar; q) = \sum_{\vec{k}} q^{|\vec{k}|} \prod_{(l,i)\neq(n,j)} \frac{a_{ln} + \hbar(k_{l,i} - k_{n,j} + j - i)}{a_{ln} + h(j - i)} \tag{13}$$

where $a_{ln} = a_l - a_n$ and it sums over partitions

$$\begin{aligned} \vec{k} &= (\vec{k}_1, \cdots, \vec{k}_N) \\ \vec{k}_l &= \{k_{l,1} \geq k_{l,2} \geq \cdots \geq k_{l,n_l} \geq k_{l,n_l+1} = \cdots = 0\} \\ |\vec{k}| &= \sum_{l,i} k_{l,i} \end{aligned} \tag{14}$$

there's also the generalization to Seiberg-Witten theory that contains matter with N_f flavors, which the partition function is

$$Z(a, m, \epsilon_1, \epsilon_2; q) = \sum_k q^k \oint_{\tilde{\mathcal{M}}_k} \mathrm{Eu}_{G\times T^2\times U(N_f)}(V \otimes M) \tag{15}$$

where $M = \mathbb{C}^{N_f}$ is the flavor space and $m = (m_1, \cdots, m_{N_f})$ are the masses, $\mathrm{Eu}_{G\times T^2\times U(N_f)}$ denotes the equivariant Euler class. Which is integrated as

$$Z(a, m, \epsilon_1, \epsilon_2; q) = \sum_{\vec{k}} \left(q\hbar^{N_f}\right)^{|\vec{k}|} \prod_{(l,i)} \prod_{f=1}^{N_f} \frac{\Gamma(\frac{a_l+m_f}{\hbar} + 1 + k_{l,i} - i)}{\Gamma(\frac{a_l+m_f}{\hbar} + 1 - i)}$$
$$\times \prod_{(l,i)\neq(n,j)} \frac{a_{ln} + \hbar(k_{l,i} - k_{n,j} + j - i)}{a_{ln} + \hbar(j - i)} \tag{16}$$

Nekrasov's Conjecture

Nekrasov noticed when $\epsilon_1 = -\epsilon_2 = \hbar$, the partition function (with or without matter) can be expanded as a genus expansion form

$$-\log Z(a, \hbar, -\hbar) = \frac{1}{\hbar^2}\mathcal{F}(a, \hbar, -\hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2}\mathcal{F}_g(a) \tag{17}$$

since the right hand side coincides the genus expansion of free energy in topological string, Nekrasov conjectured these genus expansions captures the $R^2 F^{2g-2}$ graviphoton coupling in the 4d effective action, which the topological string does in A model.

Nekrasov's Conjecture

In short, Nekrasov conjectured the instanton partition function he gives coincidence with the topological string partition function at a specified limit, which is denoted as the field theory limit in the topological string perspective. It is already proved that for the thermodynamics limit where $\epsilon_1, \epsilon_2 \to 0$, the function $\mathcal{F}(a, \epsilon_1, \epsilon_2)$ recovers the instanton part of prepotential, which is supposed to be the F_0 of the topological string theory.

Evidence for the conjecture

Historically, the evidence for the conjecture didn't come late after Nekrasov proposed it. The work of A.Iqbal and A.K.Kashani-Poor [3][4], they first showed this identification of Nekrasov partition function and Topological String partition function at $SU(2)$, where the classic version of Seiberg-Witten theory lies, later extend their result to general $SU(N)$ gauge groups. Although there is no such proof of the equivalence with mathematical rigorous, this identification is adequate for physics.

We shall go over their check of Nekrasov's conjecture in next section, following the paper of T.Eguchi and H.Kanno [8], where they proved the general agreement of this conjecture for $SU(2)$ case (note that [3][4] only done this for the field theory limit).

Evidence for the conjecture

The main tool is the geometric transition between Chern-Simons theory and topological string amplitudes (topological vertex), they used Chern-Simons invariants to describe the topological amplitudes for partition function and used various calculations to manifest its agreement with Nekrasov partition function.

As the TypeIIA side, when looking in M-theory perspective, also has a 5d version after compactification on a Calabi-Yau threefold, Nekrasov also has a similar Instanton partiton function proposal

$$Z_k^{(5D)} = \sum_{\ell_{R_1}+\ell_{R_2}=k} \prod_{\ell,m\in\{1,2\}} \prod_{i,j=1}^{\infty} \frac{\sinh R\left(a_{\ell m} + \hbar(\mu_{\ell,i} - \mu_{m,j} + j - i)\right)}{\sinh R\left(a_{\ell m} + \hbar(j-i)\right)} \tag{18}$$

where R denotes the circle where 5d greater then 4d, it is easy to see that this goes back to the 4d partition function when $R \to 0$.

Usage of the conjecture

The identification of the Nekrasov partition function and the topological string partition function is quite useful, since it gives us the topological string perspective to study field theories and vice versa.

For example if we want to study the topological string on dP_4, we have 2 methods: 1.We can write down the toric diagram of it and do a great many topological vertex calculations. 2.We can simply get the $SU(2)$ $N_f = 3$ Nekrasov partition function, using it to fix the holomorphic ambiguity from solving the HAE.

Toric diagram and topological string amplitudes

For Simplicity, in follow subsections we shall show the check of Nekrasov conjecture for $SU(2)$ for particularly $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, we shall note the general statement in the end of the section.

For some toric rational surfaces S (e.g $\mathbb{P}^2, \mathbb{F}_0, \mathbb{F}_2$), the canonical line bundle K_S over S is a non-compact Calabi-Yau manifold, which we usually called "local S". We will compute topological closed strings amplitudes on K_S.

We first note on toric diagrams. In general toric diagrams describe patterns of degeneration of the torus action. For a toric rational surface S, the diagram is made of a polygon with vertices and external lines. Along the edges of the polygon an S^1 action degenerates, and on vertices, T^2 degenerates. Our notation is to assign the i-th edge a representation R_i and Kähler parameter t_i (for trivial rep, denote as $\bullet$).

Toric diagram and topological string amplitudes

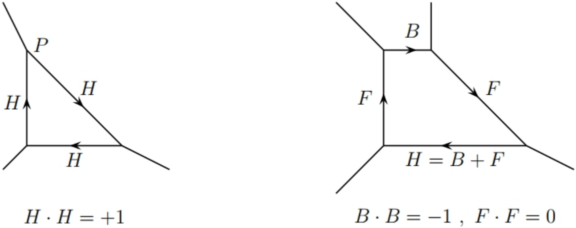

图: Toric diagram of $\mathbb{P}^2$ and $\mathbb{F}^1$

Toric diagram and topological string amplitudes

For Young diagram notations of μ^R ($\mu_1 \geq \mu_2 \geq \cdots \geq \mu_d > \mu_{d+1} = 0$),we define depth

$$d := d(\mu^R) \tag{19}$$

as the number of rows, the total number of box by

$$\ell_R := \sum_{j=1}^{d} \mu_j \tag{20}$$

and an integer κ_R by

$$\kappa_R := 2\sum_{j=1}^{d}\sum_{k=1}^{\mu_j}(k-j) = \ell_R + \sum_{j=1}^{d}\mu_j(\mu_j - 2j) \tag{21}$$

the Casimir of R is now

$$C_R = \kappa_R + N\ell_R \tag{22}$$

Toric diagram and topological string amplitudes

Based on the theory of topological vertex, the universal formula for topological closed string amplitude compactified on local Calabi-Yau manifolds K_S is

$$Z_{top\ str}^{(S)} = \sum_{R_1 \cdots R_N} W_{R_N R_1} W_{R_1 R_2} \cdots W_{R_{N-1} R_N} \cdot e^{-\sum_{i=1}^{N} t_i \cdot \ell_{R_i}} \times (-1)^{\sum_{i=1}^{N} \gamma_i \cdot \ell_{R_i}} q^{\frac{1}{2} \sum_{i=1}^{N} \gamma_i \cdot \kappa_{R_i}} \quad (23)$$

where $q = exp(2\pi i/(N+k))$ and $W_{R_i R_j}$ is the Chern-Simons invariant of the Hopf link (with the standard framing) in S^3 carring representation R_i, the "propagator" for the i-internal line is given by $e^{-t_i \cdot \ell_{R_i}}$. This is obtained by going along all the edges of the polygon.

Toric diagram and topological string amplitudes

The Chern-Simons invariant of the Hopf link is related to the topological vertex

$$C_{\bullet,R_2,R_3^t} = W_{R_2R_3} q^{-\frac{1}{2}\kappa_{R_3}} \tag{24}$$

where R^t denotes the conjugate representation obtained by exchanging row and columns of the Young diagram, according to the symmetry of the vertex

$$C_{R_1,R_2,R_3} = C_{R_2,R_3,R_1} = C_{R_3,R_1,R_2} \tag{25}$$

and the conjugation property

$$C_{R_1,R_2,R_3} = q^{\frac{1}{2}\sum_i \kappa_{R_i}} C_{R_1^t,R_3^t,R_2^t} \tag{26}$$

this implies

$$W_{R_1R_2} = q^{\frac{1}{2}\kappa_{R_2}} C_{\bullet,R_1,R_2^t} = q^{\frac{1}{2}\kappa_{R_1}} C_{\bullet,R_2,R_1^t} = W_{R_2R_1} \tag{27}$$

We introduce the Hirzebruch surface $\mathbb{F}_m$ now, which generally is a $\mathbb{P}^1$ fibre over a $\mathbb{P}^1$ base. The $H_2(\mathbb{F}_m, \mathbb{Z})$ is spanned by the two cycles B and F, and the intersection numbers are

$$B \cdot B = -m \quad F \cdot F = 0 \quad B \cdot F = 1 \tag{28}$$

it is worth noting that different m gives the $4\mathrm{d}$ gauge theory with a different theta term

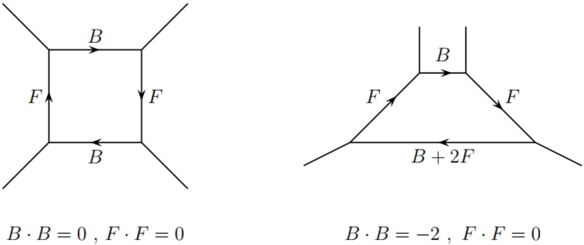

图: Toric diagram of $\mathbb{F}_0$ and $\mathbb{F}_2$

Hirzebruch surface and Topological Amplitude

the general topological string amplitude for $\mathbb{F}_m, (m = 0, 1, 2)$ is

$$Z_{top\ str}^{(\mathbf{F}_m)} = \sum_{R_1 \cdots R_4} W_{R_4 R_1} W_{R_1 R_2} W_{R_2 R_3} W_{R_3 R_4} \cdot e^{-t_F \cdot (\ell_{R_1} + \ell_{R_3} + m\ell_{R_4}) - t_B \cdot (\ell_{R_2} + \ell_{R_4})} \times (-1)^{m(\ell_{R_4} - \ell_{R_2})} q^{\frac{m}{2}(\kappa_{R_4} - \kappa_{R_2})} \tag{29}$$

where t_B and t_F are the Kähler parameters of B and F, introducing

$$K_{R_1 R_2}(Q) := \sum_S Q^{\ell_S} W_{R_1 S}(q) W_{S R_2}(q) \tag{30}$$

and the amplitude can be written as

$$Z_{top\ str}^{(\mathbf{F}_m)} = \sum_{R_1 R_2} (K_{R_1 R_2}(Q_F))^2 \cdot Q_B^{\ell_{R_1} + \ell_{R_2}} Q_F^{m\ell_{R_2}} \cdot (-1)^{m(\ell_{R_2} - \ell_{R_1})} q^{\frac{m}{2}(\kappa_{R_2} - \kappa_{R_1})} \tag{31}$$

where $Q_B := e^{-t_B}, Q_F := e^{-t_F}$.

Hirzebruch surface and Topological Amplitude

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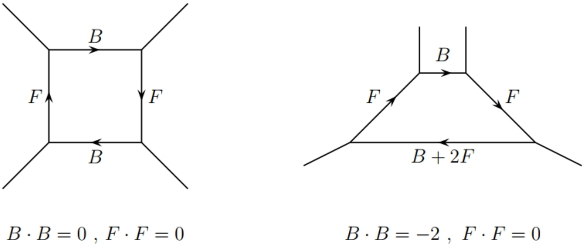

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where t_B and t_F are the Kähler parameters of B and F, introducing

$$K_{R_1 R_2}(Q) := \sum_S Q^{\ell_S} W_{R_1 S}(q) W_{S R_2}(q) \tag{34}$$

and the amplitude can be written as

$$Z_{top\ str}^{(\mathbf{F}_m)} = \sum_{R_1 R_2} (K_{R_1 R_2}(Q_F))^2 \cdot Q_B^{\ell_{R_1} + \ell_{R_2}} Q_F^{m\ell_{R_2}} \cdot (-1)^{m(\ell_{R_2} - \ell_{R_1})} q^{\frac{m}{2}(\kappa_{R_2} - \kappa_{R_1})} \tag{35}$$

where $Q_B := e^{-t_B}, Q_F := e^{-t_F}$

Iqbal and Kashani-Poor have proposed the following proposition

$$K_{R_1 R_2}(Q) = W_{R_1}(q) W_{R_2}(q) \exp\left(\sum_{n=1}^{\infty} \frac{\widetilde{f}_{R_1 R_2}(q^n)}{n} Q^n\right) \quad (36)$$

here the function $\tilde{f}_{R_1 R_2}$ is given by (recall that $[x] := q^{\frac{x}{2}} - q^{-\frac{x}{2}}, [x]_\lambda := \lambda^{\frac{1}{2}} q^{\frac{x}{2}} - \lambda^{-\frac{1}{2}} q^{-\frac{x}{2}}$)

$$\begin{aligned} \widetilde{f}_{R_1 R_2}(q) &:= W_{\square}^2(q) + f_{R_1}(q) + f_{R_2}(q) + f_{R_1}(q) f_{R_2}(q) W_{\square}^{-2}(q) \\ f_R(q) &:= \sum_{i=1}^{d} \sum_{k=1}^{\mu_i} q^{k-i} = \frac{q}{(q-1)} \sum_{i=1}^{d} \left(q^{\mu_i - i} - q^{-i}\right) \\ W_{\square}(q) &:= \frac{1}{[1]} = \frac{q^{\frac{1}{2}}}{q-1} \end{aligned} \quad (37)$$

for following discussions it is convenient to introduce the expansion coefficients $C_k(R_1, R_2)$ by

$$f_{R_1 R_2}(q) = f_{R_1}(q) + f_{R_2}(q) + (q + q^{-1} - 2) f_{R_1}(q) f_{R_2}(q) = \sum C_k(R_1, R_2) q^k$$

Hirzebruch surface and Topological Amplitude

Also Iqbal and Kashani-Poor have proved such relation

- $$\sum_k C_k(R_1, R_2) = \ell_{R_1} + \ell_3`R_2 \tag{39}$$
- $$\sum_k kC_k(R_1, R_2) = \frac{1}{2}(\kappa_{R_1} + \kappa_{R_2}) \tag{40}$$
- $$\prod_k \frac{1}{\sinh R(2a + \hbar k)^{C_k(R_1, R_2^t)}} = \prod_{i,j=1}^{\infty} \frac{\sinh R\left(2a + \hbar(\mu_{1,i} - \mu_{2,j} + j - i)\right)}{\sinh R\left(2a + \hbar(j - i)\right)} \tag{41}$$

also there is a relation expressing $K_{R_1 R_2}$ in $C_k(R_1, R_2)$

$$K_{R_1 R_2}(Q) = W_{R_1}(q) W_{R_2}(q) \exp\left(\sum_{n=1}^{\infty} \frac{W_{\square}^2(q^n)}{n} Q^n\right) \prod_k \left(1 - q^k Q\right)^{-C_k(R_1, R_2)} \tag{42}$$

Nekrasov Conjecture for 5d theory

We now would like to show the equality of Nekrasov' s formula for five dimensional gauge theory on $\mathbb{R}^4 \times S^1$ and all genus topological string amplitude for local toric Calabi-Yau manifold. More precisely, we prove that the instanton expansion of the partition function of the pure $SU(2)$ gauge theory on $\mathbb{R}^4 \times S^1$ is exactly the same as the expansion of all genus topological string amplitude for local Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ where the expansion parameter of topological string is an appropriate combination of the Kähler parameter of $\mathbb{F}_0$. Then after doing the limit reducing 5d to 4d, the desired relation is proven.

One subtlety is that only for $\mathbb{F}_0$ we can prove this relation at 5d, for similar $\mathbb{F}_1$ or $\mathbb{F}_2$ this does not prevail due to the presence of the framing factor coming from non-trivial self-intersection of the base B and the fibre F. However, they all agrees at 4d limit of the 5d theory, preserving the Nekrasov's conjecture.

Nekrasov Conjecture for 5d theory

According to Nekrasov we introduce a pair of representations of $U(N)$ denoted as R_1, R_2 in the case of $SU(2)$ gauge theory. Denoting Young tableaux as μ^{R_ℓ}, the row sequences are

$$\mu_{\ell,1} \geq \mu_{\ell,2} \geq \cdots \geq \mu_{\ell,d(\mu^{R_\ell})} > \mu_{\ell,d(\mu^{R_\ell})+1} = 0 \tag{43}$$

and the Nekrasov conjecture for the k-instanton contribution to the partition function is given by

$$Z_k^{(5D)} = \sum_{\ell_{R_1}+\ell_{R_2}=k} \prod_{\ell,m\in\{1,2\}} \prod_{i,j=1}^{\infty} \frac{\sinh R\left(a_{\ell m} + \hbar(\mu_{\ell,i} - \mu_{m,j} + j - i)\right)}{\sinh R\left(a_{\ell m} + \hbar(j-i)\right)} \tag{44}$$

where $a_{\ell,m} = a_\ell - a_m$ and we have identified the parameter R with the radius of S^1, for $SU(2)$ we have $a_{12} = -a_{21} = 2a$, the partition function is decomposed into two factors, $Z_k^{(5D,1)}$ with $\ell = m$ and $Z_k^{(5D,2)}$ with $\ell \neq m$.

Nekrasov Conjecture for 5d theory

These two can be simplified respectively

$$Z_k^{(5D,1)} = \prod_{\ell=1,2} \prod_{i\neq j}^{\infty} \frac{\sinh R\hbar(\mu_{\ell,i} - \mu_{\ell,j} + j - i)}{\sinh R\hbar(j-i)} = \prod_{\ell=1,2} \prod_{1\leq i<j<\infty} \frac{\sinh^2 R\hbar(\mu_{\ell,i} - \mu_{\ell,j} + j - i)}{\sinh^2 R\hbar(j-i)} \quad (45)$$

$$Z_k^{(5D,2)} = \prod_{i,j=1}^{\infty} \frac{\sinh^2 R\left(2a + \hbar(\mu_{1,i} - \mu_{2,j} + j - i)\right)}{\sinh^2 R\left(2a + \hbar(j-i)\right)} \quad (46)$$

Nekrasov Conjecture for 5d theory

On another side, the partition function of topological string is

$$Z^{(\mathbf{F}_m)}_{top\ str} = \exp\left(\sum_{n=1}^{\infty}\frac{2}{n}W^2_{\square}(q^n)Q_F^n\right)\sum_{R_1,R_2}Q_B^{\ell_{R_1}+\ell_{R_2}}Q_F^{m\ell_{R_2}}(-1)^{m(\ell_{R_2}-\ell_{R_1})}$$
$$\times\, q^{-\frac{m}{2}(\kappa_{R_1}+\kappa_{R_2})}W^2_{R_1}(q)W^2_{R_2^t}(q)\prod_k\left(1-q^kQ_F\right)^{-2C_k(R_1,R_2^t)} \quad (47)$$

where the factor $W_R(q)$ is the quantum dimension of the representation R or the knot invariant for an unknot in S^3 with the representation R, note although the replacement of $R \to R^t$ is made, since it sums over all representations R_2, this do not affect the result

$$W_R(q) = \dim_q R = q^{\frac{\kappa_R}{4}}\prod_{1\le i<j\le d}\frac{[\mu_i-\mu_j+j-i]}{[j-i]}\prod_{i=1}^{d}\prod_{k=1}^{\mu_i}\frac{1}{[k-i+d]}$$
$$= q^{\frac{\kappa_R}{4}}\prod_{1\le i<j<\infty}\frac{[\mu_i-\mu_j+j-i]}{[j-i]} \quad (48)$$

Nekrasov Conjecture for 5d theory

Also

$$W_R(q)^2 = 2^{-2\ell_R} q^{\kappa_R/2} \prod_{i,j}^{\infty} \frac{\sinh \frac{\beta\hbar}{2}(\mu_i - \mu_j + j - i)}{\sinh \frac{\beta\hbar}{2}(j - i)} \tag{49}$$

we note the relation

$$W_{R^t}(q) = q^{-\frac{\kappa_R}{2}} W_R(q) \tag{50}$$

We can now connect the two partitions (47) and (44), note that the first factor of (47) gives the perturbative one-loop contribution to the prepotential and the k-instanton part is identified as the sum of terms obeying the condition

$$\ell_{R_1} + \ell_{R_2} = k \tag{51}$$

as in the Nekrasov formula. We first identify the parameters as

$$q = e^{-2R\hbar}, \quad Q_F = e^{-4Ra} \tag{52}$$

Another important identity is

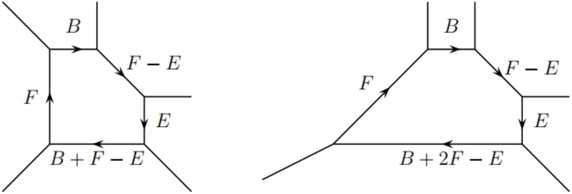
$$\prod_k \left(1 - q^k Q_F\right)^{-2C_k(R_1,R_2^t)} = (4Q_F)^{-\ell_{R_1}-\ell_{R_2}} q^{-\frac{1}{2}(\kappa_{R_1}-\kappa_{R_2})} \times \prod_k \frac{1}{[\sinh R(2a+\hbar k)]^{2C_k(R_1,R_2^t)}} \tag{53}$$

using the identities mentioned before, we can find the $\prod_k \frac{1}{[\sinh R(2a+\hbar k)]^{2C_k(R_1,R_2^t)}}$ part is identitified as $Z_k^{(5D,2)}$, only when $m = 0$ the coefficients of W_{R_1} and $W_{R_2^T}$ cancels with the factor $q^{-\frac{1}{2}(\kappa_{R_1}-\kappa_{R_2})}$.

In conclusion, the topological string partition function coincides with the Nekrasov partition function. This implies that Nekrasov’s formula encodes the entire information of topological string amplitudes, by taking suitable limits we recover four dimensional Seiberg-Witten theory or its coupling to graviphoton backgrounds.

Adding matter and Blow up

According to the prescription of geometric engineering, matters in the fundamental representation are obtained by blow ups. By making a blow up at a point on the Hirzebruch surface $\mathbb{F}_0$ we arrive the second del Pezzor surface $dP_2^{(0)}$, it has a cousin from blow up $\mathbb{F}_2$ which denoted as $dP_2^{(1)}$. Moreover, the blow up introduced a new parameter, so now we have three parameter t_F, t_B, t_E here.



$$B \cdot B = -(m+1)\ ,\ F \cdot F = 0\ ,\ B \cdot F = +1$$

$$E \cdot E = -1\ ,\ E \cdot F = E \cdot B = 0$$

Adding matter and Blow up

The topological amplitude is given by

$$Z^{(dP_2)}_{top\ str} = \sum_{R_1\cdots R_5} W_{R_5R_1}W_{R_1R_2}\cdots W_{R_4R_5}$$
$$\times\ e^{-\ell_{R_1}t_F-\ell_{R_2}t_B-\ell_{R_3}(t_F-t_E)-\ell_{R_4}t_E-\ell_{R_5}(t_B+(m+1)t_F-t_E)}$$
$$\times\ (-1)^{\ell_{R_2}+\ell_{R_3}+\ell_{R_4}}\cdot(-1)^{m(\ell_{R_5}-\ell_{R_2})}\cdot q^{-\frac{1}{2}(\kappa_{R_2}+\kappa_{R_3}+\kappa_{R_4})}\cdot q^{\frac{m}{2}(\kappa_{R_5}-\kappa_{R_2})} \quad (54)$$

the amplitude can be divided into building blocks

$$K_{R_1R_2}(Q) := \sum_S Q^{\ell_S}W_{R_1S}(q)W_{SR_2}(q)$$
$$L_{R_1R_2}(Q_1,Q_2) := \sum_{S_1,S_2} Q_1^{\ell_{S_1}}Q_2^{\ell_{S_2}}W_{R_1S_1}(q)W_{S_1S_2}(q)W_{S_2R_2}(q)$$
$$\times\ (-1)^{\ell_{S_1}+\ell_{S_2}}q^{-\frac{1}{2}(\kappa_{S_1}+\kappa_{S_2})} \quad (55)$$

Adding matter and Blow up

by cutting the diagram into two parts with the Kähler modulus t_B

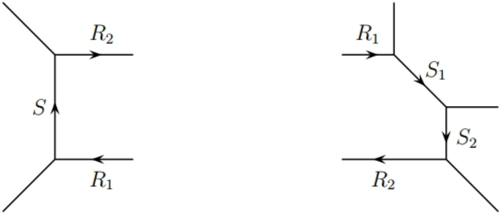


图:

the amplitude is expressed

$$Z^{(dP_2)}_{top\ str} = \sum_{R_1,R_2} Q_B^{\ell_{R_1}+\ell_{R_2}} Q_F^{(m+1)\ell_{R_2}} Q_E^{-\ell_{R_2}} (-1)^{(1-m)\ell_{R_1}+m\ell_{R_2}}$$
$$\times q^{-\frac{1}{2}((1-m)\kappa_{R_1}+m\kappa_{R_2})} K_{R_1R_2}(Q_F) \cdot L_{R_1R_2}(Q_F Q_E^{-1}, Q_E) \qquad (56)$$

Adding matter and Blow up

It is proposed that

$$L_{R_1R_2}(Q_1,Q_2) = W_{R_1}(q)W_{R_2}(q)\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}W_{\square}^2(q^n)\left((Q_1Q_2)^n - Q_1^n - Q_2^n\right)\right\} \times \prod_k \frac{\left(1-q^kQ_1\right)^{C_k(R_1)}\left(1-q^kQ_2\right)^{C_k(R_2)}}{\left(1-q^kQ_1Q_2\right)^{C_k(R_1,R_2)}} \tag{57}$$

and inserting (57)(36), one finds

$$Z_{top\ str}^{(dP_2)} = \exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}W_{\square}^2(q^n)\left(2Q_F^n - (Q_FQ_E^{-1})^n - Q_E^n\right)\right\} \times \sum_{R_1,R_2} Q_B^{\ell_{R_1}+\ell_{R_2}} Q_F^{(m+1)\ell_{R_2}} Q_E^{-\ell_{R_2}} (-1)^{(1-m)\ell_{R_1}+m\ell_{R_2}} q^{-\frac{1}{2}((1-m)\kappa_{R_1}+m\kappa_{R_2})} \times W_{R_1}^2(q)W_{R_2}^2(q)\prod_k \frac{\left(1-q^k(Q_FQ_E^{-1})\right)^{C_k(R_1)}\left(1-q^kQ_E\right)^{C_k(R_2)}}{\left(1-q^kQ_F\right)^{2C_k(R_1,R_2)}} \tag{58}$$

Adding matter and Blow up

The first factor of $Z^{dP_2}_{top\ str}$ is identified as the perturbative part. For the perturbative part, if we do the identification of parameters as

$$Q_F = e^{-4Ra} \quad Q_E = e^{-2R(a-m)} \quad Q_F Q_E^{-1} = e^{-2R(a+m)} \tag{59}$$

and

$$q = e^{-2\hbar R} \tag{60}$$

when $q \to 0$, we have $W^2_{\square}(q^n) = \frac{1}{\left(e^{n\hbar R} - e^{-n\hbar R}\right)^2} \to (2n\hbar R)^{-2}$ therefore

$$\mathcal{F}_{one\ loop} \to \frac{1}{4R^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \left(2e^{-4nRa} - e^{-2nR(a+m)} - e^{-2nR(a-m)}\right) \tag{61}$$

Adding matter and Blow up

we can calculate

$$\tau_{one\ loop} = \frac{\partial^2 \mathcal{F}_{one\ loop}}{\partial a^2}$$
$$= -8\log(1-e^{-4nRa}) + \log(1-e^{-2nR(a+m)})(1-e^{-2nR(a-m)}) \quad (62)$$

which is

$$\tau_{one\ loop} = -8\log\sinh 2Ra + \log\sinh R(a+m) + \log\sinh R(a-m) \quad (63)$$

which gives the correct behavior of the coupling constant when $R \to \infty$

$$\tilde{\tau} = \frac{\tau}{R} = -16|a| + |a+m| + |a-m| \quad (64)$$

The instanton partition also can be simplified using the relation $C_k(R^t) = C_{-k}(R)$ and by the similar calculation performed in (48)

$$Z^{(dP_2)}_{top\ str} = \sum_{R_1,R_2} \left(\frac{Q_B}{8Q_F}\right)^{\ell_{R_1}+\ell_{R_2}} Q_F^{2\ell_{R_2}+\frac{1}{2}\ell_{R_1}} Q_E^{-\frac{1}{2}(\ell_{R_1}+\ell_{R_2})} q^{\frac{1}{4}(\kappa_{R_1}+\kappa_{R_2})}$$

$$\times \prod_k (\sinh R(a+m+\hbar k))^{C_k(R_1)} (\sinh R(-a+m+\hbar k))^{C_k(R_2)}$$

$$\times \prod_{\ell,m\in\{1,2\}} \prod_{i,j=1}^{\infty} \frac{\sinh R\left(a_{\ell m} + \hbar(\mu_{\ell,i} - \mu_{m,j} + j - i)\right)}{\sinh R\left(a_{\ell m} + \hbar(j-i)\right)} \qquad (65)$$

in the 4d limit, the last two factors are

$$\frac{1}{R^{3(\ell_{R_1}+\ell_{R_2})}} \prod_k (a+m+\hbar k)^{C_k(R_1)} (-a+m+\hbar k)^{C_k(R_2)}$$

$$\times \prod_{\ell,m\in\{1,2\}} \prod_{i,j=1}^{\infty} \frac{a_{\ell m} + \hbar(\mu_{\ell,i} - \mu_{m,j} + j - i)}{a_{\ell m} + \hbar(j-i)} \qquad (66)$$

Adding matter and Blow up

by the definition that $C_k(R)$'s generating function is

$$f_R(q) = \sum_{i=1}^{d(\mu^R)} \sum_{j=1}^{\mu_i} q^{j-i} \tag{68}$$

we can identify the first line of (67) is

$$\prod_{i=1}^{d(\mu^{R_1})} \prod_{j=1}^{\mu_{1,i}} (a + m + \hbar(j-i)) \times \prod_{i=1}^{d(\mu^{R_2})} \prod_{j=1}^{\mu_{2,i}} (-a + m + \hbar(j-i)) \tag{69}$$

which shows that the two partition function agrees eventually.

The method can also be generalized to general $SU(N)$ case, toric diagram is rather complicated

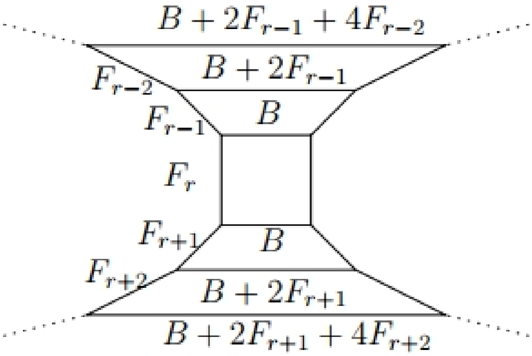


图: $SU(N)$ Toric diagram

Short note for $SU(N)$

And the 5d Nekrasov instanton partition function is given

$$Z_{Nek}^{(5D)} = \sum_{R_{1,\cdots,N}} \varphi^{l_{R_1}+\cdots+l_{R_N}} \prod_{l,n=1}^{N} \prod_{i,j=1}^{\infty} \frac{\sinh R(a_{ln} + \hbar(\mu_{l,i} - \mu_{n,i} + j - i))}{\sinh R(a_{ln} + \hbar(j-i))} \tag{70}$$

Generally, the two main identities are still used to simplify the vertex calculation

$$W_R(q)^2 = 2^{-2\ell_R} q^{\kappa_R/2} \prod_{i,j}^{\infty} \frac{\sinh \frac{\beta\hbar}{2}(\mu_i - \mu_j + j - i)}{\sinh \frac{\beta\hbar}{2}(j-i)}$$

$$\prod_k \left(1 - q^k Q_F\right)^{-2C_k(R_1,R_2^t)} = (4Q_F)^{-\ell_{R_1}-\ell_{R_2}} q^{-\frac{1}{2}(\kappa_{R_1}-\kappa_{R_2})} \times \prod_k \frac{1}{[\sinh R(2a + \hbar k)]^{2C_k(R_1,R_2^t)}} \tag{71}$$

Short note for $SU(N)$

It is proved that the partition function agrees the instanton partition function given from geometric transition after identifying

$$\varphi = \frac{Q_B}{2^{2N} D(Q_{F_i})} \tag{72}$$

$$D(Q_{F_i}) = \begin{cases} \prod_{i=1}^{\frac{N-1}{2}} Q_{F_i}^{i} \prod_{i=\frac{N-1}{2}+1}^{N-1} Q_{F_i}^{N-i} & , N = \text{odd} \\ \prod_{i=1}^{\frac{N}{2}-1} Q_{F_i}^{i} \prod_{i=\frac{N}{2}+1}^{N-1} Q_{F_i}^{N-i} & , N = \text{even.} \end{cases}$$

and taking the field theory limit

$$Q_B = (-1)^{N-m} (\frac{\beta \Lambda}{2})^{2N} \, , \; Q_{F_j} = e^{-\beta a_{j,j+1}} \, , \; \beta \to 0 \tag{73}$$

We have mentioned that (17) when at a special limit, the standard form of genus expansion can be obtained from Nekrasov Instanton Partition function, as the original hint of connecting Topological Strings Partition function and Nekrasov Instanton Partition function. However, for the generalized version of Nekrasov Instanton Partition function (12), we also has a similar expansion by two parameter

$$\log Z(a,\epsilon_1,\epsilon_2)=\sum_{i,j=0}^{\infty}(\epsilon_1+\epsilon_2)^i(\epsilon_1\epsilon_2)^{j-1}F^{(\frac{i}{2},j)}(a) \tag{74}$$

From the relation between M theory and A model topological string theory, the BPS state has a quantum number of

$$\left[\left(\frac{1}{2},0\right)+2\,(0,0)\right]\otimes(j_L,j_R) \tag{75}$$

and the number of such states are counted by $n^{\vec{d}}_{j_L,j_R}$, for $\vec{d}$ denotes the 2 cycle where the M2 brane wrapped.

The BPS partition function is

$$Z_{BPS} = \mathrm{Tr}_{\mathcal{H}_{BPS}}(-1)^{2(m_L+m_R)} q_L^{2m_L} q_R^{2m_R} e^{-\vec{t}\cdot\vec{d}} \tag{76}$$

where

$$q_L = exp(-\frac{\epsilon_1 - \epsilon_2}{2}) \quad q_R = exp(-\frac{\epsilon_1 + \epsilon_2}{2}) \tag{77}$$

and we can give a physical definition of the refined theory

$$Z_{top}(\vec{t}, \epsilon_1, \epsilon_2) = Z_{BPS}(\vec{t}, \epsilon_1, \epsilon_2) \tag{78}$$

Still, we have two methods on refined topological string, refined topological vertex for A model and refined holomorphic anomaly equation for B model.

This Refined version of topological string partition function has several interesting limits

- For $\epsilon_1 = -\epsilon_2 = ig_s$, this becomes the usual genus expansion of topological sting
$$\sum_{g=0}^{\infty} g_s^{2g-2} F^{(0,g)}(a) \tag{79}$$
- For $\epsilon_1 = 0, \epsilon_2 = \hbar \neq 0$, only $j = 1$ contributes. This is called the Nekrasov-Shatashvili limit, this closely relates to integrable systems.
- If we define
$$\epsilon_1 = g_s\sqrt{\beta} \quad \epsilon_2 = \frac{g_s}{\sqrt{\beta}} \tag{80}$$
the Alday-Gaiotto-Tachikawa conjecture claims the equivalence to Liouville conformal field theory at central charge
$$c = 1 + 6Q^2 \quad Q = \beta^{\frac{1}{2}} + \beta^{-\frac{1}{2}} \tag{81}$$

Refined Holomorphic Anomaly Equation

We first note for $F^{(\frac{i}{2},j)}$, by this notation, this goes back to F_g at $F^{(0,g)}$. Normally when i is an odd integer, $F^{(\frac{i}{2},j)}$ vanishes except a trivial term from the pertubative contributions, for most models we only need to concern $F^{(i,j)}$, but for some (namely $SU(2)$ $N_f = 1, 2, 3$), also $F^{(\frac{1}{2},0)}$ contributes.

The generalized HAE of refined topological string is

$$\bar{\partial}_{\bar{i}} F^{(g_1,g_2)} = \frac{1}{2} \bar{C}_{\bar{i}}^{jk} \left(D_j D_k F^{(g_1,g_2-1)} + {\sum_{r_1,r_2}}' D_j F^{(r_1,r_2)} D_k F^{(g_1-r_1,g_2-r_2)} \right) \quad (82)$$

where ${\sum_{r_1,r_2}}'$ denotes that $(r_1, r_2) = (0, 0), (g_1, g_2)$ is not included. When $g_1 = 0$, this reduces to ordinary HAE.

One example of direct integration method [13] is introduced in the HAE seminar.

We know that the topological vertex

$$C_{\lambda\mu\nu} = q^{\frac{\kappa(\mu)}{2}} s_{\nu^t}(q^{-\rho}) \sum_{\eta} s_{\lambda^t/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu^t-\rho}) \quad s_{\mu/\nu} := \sum_{\lambda} c^{\mu}_{\nu\lambda} s_{\lambda}(x) \tag{83}$$

Historically, by generalizing the 3D partition interpretation of topological vertex, the refined version is defined[14].

$$C_{\lambda,\mu,\nu}(t,q) = \left(\frac{q}{t}\right)^{\frac{||\mu||^2+||\nu||^2}{2}} t^{\frac{\kappa(\mu)}{2}} P_{\nu^t}(t^{-\rho};q,t) \sum_{\eta} \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu} \tag{84}$$

In the above expression, P_{ν} is the Macdonald function

$$\begin{aligned} P_{\nu^t}(t^{-\rho};q,t) &= t^{\frac{||\nu||^2}{2}} \tilde{Z}_{\nu}(t,q) \\ \tilde{Z}_{\nu}(t,q) &= \prod_{(i,j)\in\nu} (1-t^{a(i,j)+1}q^{\ell(i,j)})^{-1}, \quad a(i,j) = \nu^t_j - i, \quad \ell(i,j) = \nu_i - j \end{aligned} \tag{85}$$

Refined Topological Vertex

One biggest difference is the refined vertex are not cyclically symmetric, also, there are two different parameters q, t present. Since in unrefined case that $q = e^{ig_s}$, we can think that $q = e^{i\epsilon_1}, t = e^{i\epsilon_2}$ now. Normally the unrefined topological vertex gives the A model partition, which is the generating function of GW invariants. The refined vertex gives the refined invariants, such as Khovanov Knot invariants.

Since the cyclic symmetry disappeared, the refined topological vertex has one preferred direction, what's the imapct? Since we can assign Lagrangian submanifolds (D-branes) to edges of the vertex, the three different assignment has different results.

Refined Topological Vertex

We consider the three different vertex

$$C_{\lambda 00},\ C_{0\mu 0},\ C_{00\nu} \tag{86}$$

It turns out that the partition function of assigning D branes to the first two are same (x below are the eigenvalues of the holonomy matrix, this is noted as a scalar because only one brane persists now, there are only one nontrivial eigenvalues)

$$\begin{aligned} Z(t,q,x)_1 &= \sum_\lambda C_{\lambda 00}(t^{-1},q^{-1}) s_\lambda(x) = \prod_{i=1}^{\infty}(1 - Qt^{-i+\frac{1}{2}}) \\ Z(t,q,x)_1 &= \sum_\mu C_{0\mu 0}(t^{-1},q^{-1}) s_\mu(x) = \prod_{i=1}^{\infty}(1 - Qq^{-i+\frac{1}{2}}) \end{aligned} \tag{87}$$

But not for the third direction

$$Z(t,q,x) = \sum_\nu C_{00\nu}(t^{-1},q^{-1}) s_\nu(-Q) = \sum_{k=0}^{\infty}\left(\frac{Qt}{\sqrt{k}}\right)^k \prod_{n=1}^{k}(1 - tq^{n-1})^{-1} \tag{88}$$

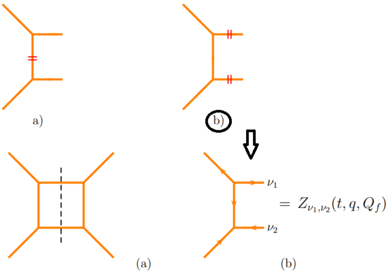
This give rise to the correction of the nonrefined framing factor $q^{-\frac{\kappa(\mu)}{2}}$ on preferred directions, we have

$$f_\nu(t,q) := (-1)^{|\nu|} t^{n(\nu)} q^{-n(\nu^t)} = (-1)^{|\nu|} \left(\frac{t}{q}\right)^{n(\nu)} q^{-\frac{\kappa(\nu)}{2}} \tag{89}$$

which is obtained by 3d partitions, this clearly recovers the nonrefined framing factor when $q = t$.

We can now move our sight to the Local $\mathbb{P}^1 \times \mathbb{P}^1$ geometry, for simplicity, we shall not go deep into details. The result is that the refined topological vertex actually recovers the full Nekrasov partition function. For the process of vertex calculation side is following

Refined Topological Vertex



$$Z(Q_b,Q_f,t,q) := \sum_{\nu_1,\nu_2} (-Q_b)^{|\nu_1|+|\nu_2|} Z_{\nu_1,\nu_2}(t,q,Q_f)\, f_{\nu_1}(t,q)\, f_{\nu_2}(q,t)\, Z_{\nu_2,\nu_1}(q,t,Q_f),$$

$$Z_{\nu_1,\nu_2}(t,q,Q_f) \;=\; q^{\frac{||\nu_1||^2}{2}+\frac{||\nu_2^t||^2}{2}} \widetilde{Z}_{\nu_1}(t,q)\widetilde{Z}_{\nu_2^t}(t,q) \prod_{i,j} \left(1-Q_f\, t^{i-1-\nu_{2,j}}\, q^{j-\nu_{1,i}}\right)^{-1}.$$

⇓

$$Z_{inst}(Q_b,Q_f,t,q) \;=\; \sum_{\nu_1,\nu_2} Q_b^{|\nu_1|+|\nu_2|}\, q^{||\nu_2^t||^2}\, t^{||\nu_1^t||^2}\, \widetilde{Z}_{\nu_1}(t,q)\widetilde{Z}_{\nu_2^t}(t,q)\, \widetilde{Z}_{\nu_2}(q,t)\widetilde{Z}_{\nu_1^t}(q,t)$$

$$\times \prod_{i,j=1}^{\infty} \frac{(1-Q_f\, t^{i-1}q^j)(1-Q_f\, q^{i-1}t^j)}{(1-Q_f\, t^{i-1-\nu_{2,j}}q^{j-\nu_{1,i}})(1-Q_f\, q^{i-1-\nu_{1,j}}t^{j-\nu_{2,i}})}$$

Refined Topological Vertex

Which matches the gauge theory side

$$
\begin{aligned}
Z^{U(2)}_{\text{gauge theory}} &= \sum_{\nu_1,\nu_2} \mathfrak{q}^{|\nu_1|+|\nu_2|} Z(\nu_1^t,\nu_2^t;Q,t,q) \\
Z(\nu_1,\nu_2;Q,t,q) &:= \left(\frac{q}{t}\right)^{|\nu_1|+|\nu_2|} q^{||\nu_1^t||^2}\, t^{||\nu_2||^2}\, \widetilde{Z}_{\nu_1^t}(t,q)\widetilde{Z}_{\nu_1}(q,t)\widetilde{Z}_{\nu_2^t}(t,q)\widetilde{Z}_{\nu_2}(q,t)\, G(\nu_1,\nu_2,Q,t,q) \\
G(\nu_1,\nu_2,Q,t,q) &= \prod_{i,j=1}^{\infty} \frac{(1-Q\,q^{j-1}t^{i})(1-Q\,q^{j}t^{i-1})}{(1-Q\,q^{-\nu^t_{2,i}+j-1}\,t^{-\nu_{1,j}+i})(1-Q\,q^{-\nu^t_{2,i}+j}\,t^{-\nu_{1,j}+i-1})}
\end{aligned}
$$

$$
\widetilde{Z}_\nu(t,a) = \prod_{s\in\nu}(1-t^{a(s)+1}q^{\ell(s)})^{-1}\,,\quad \ell(i,j)=\nu_i-j\,,\quad a(i,j)=\nu^t_j-i
$$

Topological String and $4\mathrm{d}$ $\mathcal{N}=1$ theories

We have considered the relation between topological string and $\mathcal{N}=2$ theories, now we turn to $\mathcal{N}=1$.

The breaking of supersymmetry from $\mathcal{N}=2$ to $\mathcal{N}=1$ has two major ways. The first one adds D-brane to the four uncompactified dimensions, the second one adds flux to the six Calabi-Yau dimensions. At the geometric transition aspect, these two ways are equivalent at some sense. In $\mathcal{N}=1$ theories, superpotential plays a even more important role, since they determine the IR physics as the prepotential does in $\mathcal{N}=2$. We shall see how topological strings compute superpotential here.

Breaking to $\mathcal{N} = 1$ with branes

Beginning with TypeII string theory on a Calabi-Yau X, we get $\mathcal{N} = 2$ supersymmetry at $4\mathrm{d}$. Then introduce N D-branes wrapping cycles in the Calabi-Yau with fill the spacetime dimensions, we reduce the supersymmetry to $\mathcal{N} = 1$.

What's the connection between topological string on X and $4\mathrm{d}$ gauge theory? Firstly, we have open topological strings in this configuration, once organize $F_{0,h}$ by

$$F(S) = \sum_{h=0}^{\infty} F_{0,h} S^h \tag{90}$$

Breaking to $\mathcal{N} = 1$ with branes

It is known that the F-term of the $\mathcal{N} = 1$ theory is also computed by the topological string theory, which is written as

$$\int d^4x \int d^2\theta \left(N \frac{\partial F}{\partial S} + \tau S \right) \tag{91}$$

where S is regarded as the glueball of $4\mathrm{d}$ theory, with lowest component $\mathrm{Tr}(\psi_\alpha \psi^\alpha)$. Also the Yang-Mills coupling is included $\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi}$, this totally gives the glueball superpotential,

$$W(S) = N \frac{\partial F}{\partial S} + \tau S \tag{92}$$

capaturing the infrared dynamics, by describing the condensation to glueball at $W'(S) = 0$.

Breaking to $\mathcal{N} = 1$ with fluxes

We consider the TypeIIB superstring on Calabi-Yau X, which has the prepotential term of

$$\int d^4x \int d^4\theta F_0(X^I) \tag{93}$$

If we introduce N^I units of Ramond-Ramond 3form flux on the $I - th$ A-cycle of the Calabi-Yau, in $\mathcal{N} = 2$ supergravity language, it turns out that this corresponds to the θ^2 component of X^I, the F-term reads

$$\int d^4x \int d^2\theta N^I \frac{\partial F_0}{\partial X^I} \tag{94}$$

for superpotential, this is interpreted [10], where the extension to include flux τ_I on the $I - the$ B cycle is also written

$$W(X^I) = N^I \frac{\partial F_0}{\partial X^I} + \tau_I X^I \tag{95}$$

Geometric Transition of branes and fluxes

The two different approaches above can be related in the aspect of geometric transition [11]. Starting with a Calabi-Yau which has a nontrivial 2cycle, then wrap N D5-brane on this 2cycle, obtaining a $U(N)$ gauge theory. There is a dual geometry where the D5-branes disappear and are replaced by a 3-cycle A, in this dual geometry there are N units of Ramond-Ramond flux on the dual cycle B. The claim is that the physical string theories on these two geometries are equivalent in the IR, after we identify the glueball superfield S with the period Ω over the A cycle in the dual geometry. Of course one notice the conifold transition is a special case.

Geometric Transition of branes and fluxes

Take the conifold transition as an example, at the brane picture, we wrap D5-branes on $\mathbb{P}^1$ of the resolved conifold. The dual geometry after the transition is the deformed conifold, which has a compact S^3 and its dual B cycle, with corresponding periods

$$X = \int_A \Omega = t$$
$$F = \int_B \Omega = t \log t \tag{96}$$

and the prepotential is

$$F_0 = \frac{1}{2} t^2 \log t \tag{97}$$

to compare with the gauge theory, we identify $t = S$, leads to the superpotential

$$W(S) = NS \log S - 2\pi i \tau S \tag{98}$$

by extremizing $W(S)$, one finds the expected vacua of $\mathcal{N} = 1$ SYM

$$S = \Lambda_0^3 exp(2\pi i j \tau / N) \tag{99}$$

Topological Strings and BPS states

One particular interesting thing is the elliptic genus of the $5d\ \mathcal{N}=2$ theory

$$\mathrm{Tr}(-1)^{J_R} q^{J_L} e^{-\beta H} \tag{100}$$

it is independent of the complexstructure moduli of X, although it can and does depend continuously on the Kähler moduli. This property is reminiscent of the A model topological string, and indeed it turns out that the A model partition function is precisely the elliptic genus by identifying

$$q = e^{-g_s} \tag{101}$$

Such a connection seems reasonable: after all, the A model counts holomorphic maps, and the image of a holomorphic map is a supersymmetric cycle on which a brane could be wrapped to give a BPS state.

Topological Strings and BPS states

This connection actually means that the BPS counting's dependence of spin are similar with the topological string's dependence of worldsheet genus. Recall what the Schwinger calculation told us, when we tried to integrate out the charged particles (from D2 and D0 branes) contribution to the R_+^2 correction ar self dual graviphoton background, which actually coincides the 4d F term contribution from topological string computation

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-n\langle Q,t\rangle}}{(2\ sinh \frac{nF_+}{2})^2} = \sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-n\langle Q,t\rangle}}{(2\ sinh \frac{ng_s}{2})^2} \tag{102}$$

This equivalence inspired people to express the topological string free energy in BPS degeneracies

$$F(t, g_s) = \sum_{j\geq 0} \sum_{Q\in H_2(X,\mathbb{Z})} \mathcal{N}_{j,Q} (\sum_{n\geq 0} (2\ sinh \frac{ng_s}{2})^{2j-2} e^{-n\langle Q,t\rangle}) \tag{103}$$

which is the Gopakumar-Vafa formalism.

Topological Strings and BPS states

To reproduce this, one needs a 5d hypermultiplet field ϕ, choosing complex coordinates z_1, z_2 on $\mathbb{R}^4$, we can write

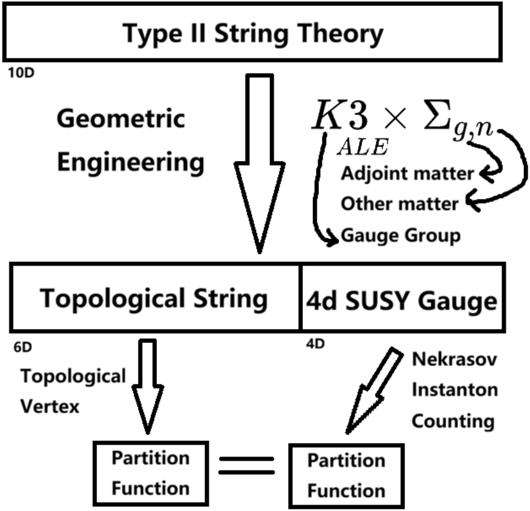
$$\phi = \phi(z_1, z_2, \bar{z}_1, \bar{z}_2) \tag{104}$$

and the BPS excitations are the ones independent of $\bar{z}_i$

$$\phi = \sum_{l,m \geq 0} \phi_{lm} z_1^l z_2^m \tag{105}$$

By expanding, we get a collection of creation operators ϕ_{lm}. The operator ϕ_{lm} creates $SU(2)_L$ spin $l + m + 1$, so there are n of them that create spin n and BPS mass $\langle Q, t \rangle$, and the second quantization of these operators then accounts for the factor $\prod_{n=1}^{\infty}(1 - q^n e^{-\langle Q,t \rangle})^n$.

Summary



Type II String Theory
10D
Geometric
Engineering
$K3 \times \Sigma_{g,n}$
ALE
Adjoint matter
Other matter
Gauge Group
Topological String
4d SUSY Gauge
6D
4D
Topological
Vertex
Nekrasov
Instanton
Counting
Partition
Function
Partition
Function

Thanks for listening!

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